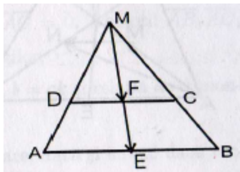
5.2 Coliniaritatea a doi vectori . Exerciții

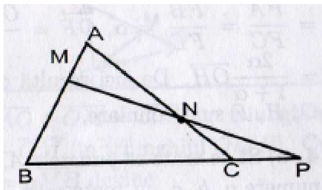
1) Se consideră ΔABC și M, N, P astfel încât $\overrightarrow{AM} = \frac{2}{3}\overrightarrow{AB}$, $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$, $\overrightarrow{AC} = \overrightarrow{CP}$. Arătați că : a) $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC} - \frac{1}{3}\overrightarrow{BA}$, $\overrightarrow{NP} = \frac{3}{2}\overrightarrow{BC} - \overrightarrow{BA}$; b) M, N, P sunt coliniare .

2) În paralelogramul $ABCD$, se consideră punctele E și F pe $[AD]$ și respectiv $[CD]$, astfel încât $\overrightarrow{AE} = \overrightarrow{ED}$ și $\frac{CF}{FD} = \frac{1}{3}$. Arătați că vectorii $2\overrightarrow{CE} + \overrightarrow{AF}$ și $\overrightarrow{AB}$ sunt coliniari .

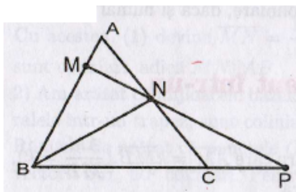
3) Fie $ABCD$ un trapez cu $AB \parallel CD$, $AD \cap BC = \{M\}$, iar E și F mijloacele bazelor $[AB]$ și respectiv $[CD]$. Arătați că punctele M, F, E sunt coliniare .



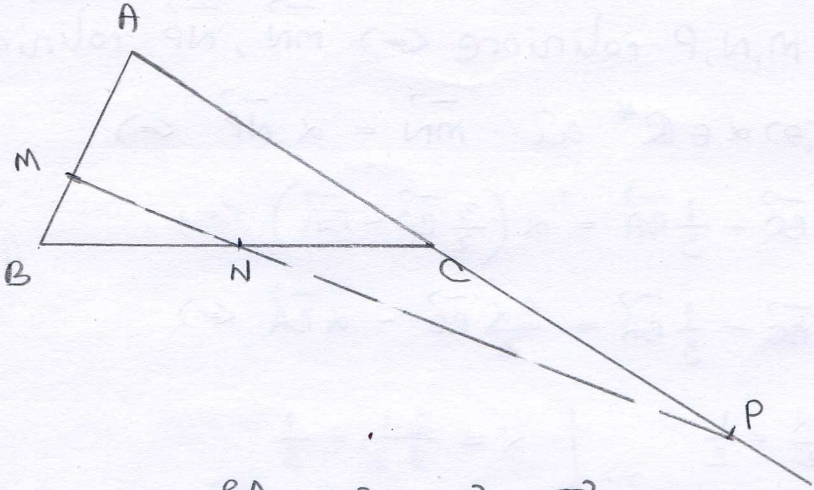
4) Pe laturile $[AB]$, $[BC]$, $[CA]$ ale triunghiului ΔABC se consideră punctele M, N și respectiv P, astfel încât $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{MB}$, $\overrightarrow{PB} = 5\overrightarrow{PC}$ și $\overrightarrow{CN} = \frac{3}{5}\overrightarrow{NA}$. Arătați că punctele M, N, P sunt coliniare .



5) Fie ΔABC, iar $M \in [AB]$, $N \in [AC]$, $P \in BC$, $C \in [BP]$ astfel încât $\frac{MA}{MB} = \frac{1}{3}$, $\frac{NC}{NA} = \frac{2}{3}$, $\frac{PC}{PB} = \frac{2}{9}$. Să se arate că punctele M, N, P sunt coliniare .



Coliniaritatea a 2 vectori



) în $\Delta BNM \overset{R.\Delta}{\Rightarrow} \vec{MN} = \vec{MB} + \vec{BN}$ (1)

$$\vec{MB} = \vec{MA} + \vec{AB} = -\vec{AM} + \vec{AB} = -\frac{2}{3}\vec{AB} + \vec{AB} =$$

$$= \frac{-2\vec{AB} + 3\vec{AB}}{3} = \frac{1}{3}\vec{AB} = -\frac{1}{3}\vec{BA} \quad (2)$$

$$\vec{BN} = \frac{1}{2}\vec{BC} \quad (3)$$

1,2,3 $\Rightarrow \vec{MN} = -\frac{1}{3}\vec{BA} + \frac{1}{2}\vec{BC}$

în $\Delta CNP \overset{R.\Delta}{\Rightarrow} \vec{NP} = \vec{NC} + \vec{CP}$ (4)

$$\vec{NC} = \vec{NB} + \vec{BC} = -\vec{BN} + \vec{BC} = -\frac{1}{2}\vec{BC} + \vec{BC} =$$

$$\frac{-\vec{BC} + 2\vec{BC}}{2} = \frac{1}{2}\vec{BC} \quad (5)$$

$$\vec{CP} = \vec{AC} = \vec{AB} + \vec{BC} = -\vec{BA} + \vec{BC} \quad (6)$$

,5,6 $\Rightarrow \vec{NP} = \frac{1}{2}\vec{BC} - \vec{BA} + \vec{BC} = \frac{\vec{BC} + 2\vec{BC}}{2} - \vec{BA} =$

$\frac{3}{2}\overrightarrow{BC} - \overrightarrow{BA}$ ■

) M, N, P coliniare $\Leftrightarrow$ $\overrightarrow{MN}$, $\overrightarrow{NP}$ coliniari

$\Leftrightarrow (\exists)\, \alpha \in \mathbb{R}^*$ a.î. $\overrightarrow{MN} = \alpha \cdot \overrightarrow{NP}$ $\Leftrightarrow$

$\frac{1}{2}\overrightarrow{BC} - \frac{1}{3}\overrightarrow{BA} = \alpha\left(\frac{3}{2}\overrightarrow{BC} - \overrightarrow{BA}\right)$ $\Leftrightarrow$

$\frac{1}{2}\overrightarrow{BC} - \frac{1}{3}\overrightarrow{BA} = \frac{3\alpha}{2}\overrightarrow{BC} - \alpha\overrightarrow{BA}$ $\Leftrightarrow$

$$\begin{cases} \frac{3\alpha}{2} = \frac{1}{2} \\ -\alpha = -\frac{1}{3} \end{cases} \Rightarrow \begin{cases} \alpha = \frac{\cancel{2}\cdot 1}{3\cdot\cancel{2}} = \frac{1}{3} \\ \alpha = \frac{1}{3} \end{cases} \Rightarrow$$

M, N, P coliniare. ■

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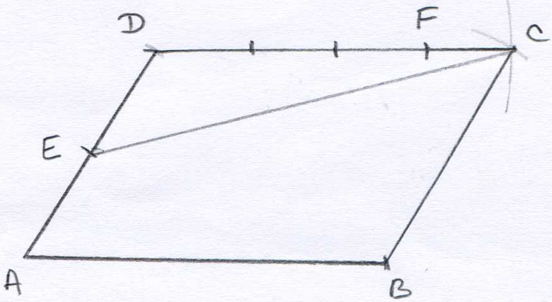


$\vec{AE} = \vec{ED} \Rightarrow E$ mij $[AD] \Rightarrow \vec{CE} = \dfrac{\vec{CA} + \vec{CD}}{2} \,/\cdot$

$\Rightarrow 2\vec{CE} = \vec{CA} + \vec{CD}$ (1)

$\dfrac{CF}{FD} = \dfrac{1}{3} = k \Rightarrow \vec{AF} = \dfrac{{}^{3/}\vec{AC} + \frac{1}{3}\vec{AD}}{{}^{3/}1 + \frac{1}{3}} \Rightarrow$

$\Rightarrow \vec{AF} = \dfrac{3\vec{AC} + \vec{AD}}{\cancel{3}} \cdot \dfrac{\cancel{3}}{4} = \dfrac{3\vec{AC} + \vec{AD}}{4}$ (2)

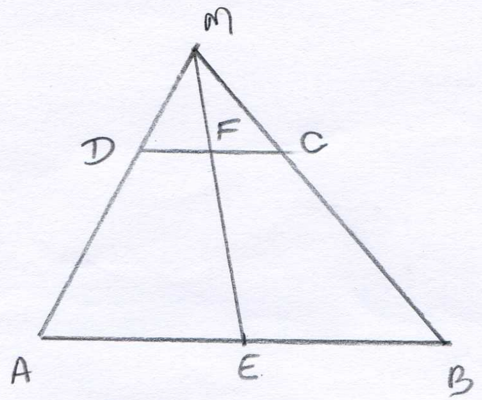
$1,2 \Rightarrow 2\vec{CE} + \vec{AF} = {}^{4/}\vec{CA} + {}^{4/}\vec{CD} + \dfrac{3\vec{AC} + \vec{AD}}{4} =$

$= \dfrac{4\vec{CA} + 4\vec{CD} + 3\vec{AC} + \vec{AD}}{4} = \dfrac{4\vec{CA} - 3\vec{CA} + 4\vec{C}}{4}$

$= \dfrac{\vec{CA} + \vec{AD} + 4\vec{CD}}{4} = \dfrac{\vec{CD} + 4\vec{CD}}{4} = \dfrac{5\vec{CD}}{4} = \dfrac{-5}{4}$

$\Rightarrow 2\vec{CE} + \vec{AF}$ și $\vec{AB}$ sunt coliniari ■

5)



$CD \parallel AB \xrightarrow{T.T.} \frac{MD}{DA} = \frac{MC}{CB} \Rightarrow \frac{MD}{MD+DA} = \frac{M}{M}$

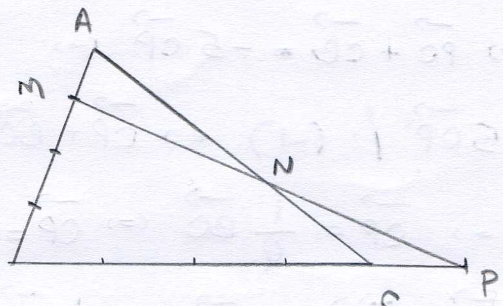
$\Rightarrow \frac{MD}{MA} = \frac{MC}{MB} = k \Rightarrow \begin{cases} \overrightarrow{MD} = k \cdot \overrightarrow{MA} \\ \overrightarrow{MC} = k \cdot \overrightarrow{MB} \end{cases}$ (1)

F mij. $[CD] \Rightarrow \overrightarrow{MF} = \frac{\overrightarrow{MD} + \overrightarrow{MC}}{2}$ (2)

E mij $[AB] \Rightarrow \overrightarrow{ME} = \frac{\overrightarrow{MA} + \overrightarrow{MB}}{2}$ (3)

$1,2 \Rightarrow \overrightarrow{MF} = \frac{k\overrightarrow{MA} + k\overrightarrow{MB}}{2} = k \cdot \frac{\overrightarrow{MA} + \overrightarrow{MB}}{2} \overset{(3)}{=}$

$= k \cdot \overrightarrow{ME} \Rightarrow \overrightarrow{MF}, \overrightarrow{ME}$ coliniari $\Rightarrow M, F, E$



$(\vec{AB}, \vec{AC})$ – bază

$\Delta AMN \overset{R.\Delta}{\Rightarrow} \vec{MN} = \vec{MA} + \vec{AN}$ (1)

$\vec{AM} = \frac{1}{3}\vec{MB} \Leftrightarrow -\vec{MA} = \frac{1}{3}(\vec{MA} + \vec{AB}) \Leftrightarrow$

$-\vec{MA} = \frac{1}{3}\vec{MA} + \frac{1}{3}\vec{AB} \Rightarrow \frac{1}{3}\vec{MA} + \vec{MA} = -\frac{1}{3}$

$\Rightarrow \frac{4}{3}\vec{MA} = -\frac{1}{3}\vec{AB} \;\Big|\cdot\frac{3}{4} \Rightarrow \vec{MA} = -\frac{1}{4}\vec{AB}$ (

$\vec{CN} = \frac{3}{5}\vec{NA} \Leftrightarrow \vec{CA} + \vec{AN} = -\frac{3}{5}\vec{AN} \Leftrightarrow$

$\vec{AN} + \frac{3}{5}\vec{AN} = -\vec{CA} \Leftrightarrow \frac{8}{5}\vec{AN} = \vec{AC} \;\Big|\cdot\frac{5}{8}$

$\Rightarrow \vec{AN} = \frac{5}{8}\cdot\vec{AC}$ (3)

$1,2,3 \Rightarrow \vec{MN} = -\frac{1}{4}\vec{AB} + \frac{5}{8}\vec{AC}$ (*)

$\Delta NCP \overset{R.\Delta}{\Rightarrow} \vec{NP} = \vec{NC} + \vec{CP}$ (4)

$\vec{CN} = \frac{3}{5}\vec{NA} \Leftrightarrow -\vec{NC} = -\frac{3}{5}\vec{AN} \Rightarrow \vec{NC} = \frac{3}{5}\vec{AN}$

$\Rightarrow \vec{NC} = \frac{3}{5}\cdot\frac{5}{8}\vec{AC} \Rightarrow \vec{NC} = \frac{3}{8}\vec{AC}$ (5)

$\overrightarrow{PB} = 5\overrightarrow{PC} \Leftrightarrow \overrightarrow{PC} + \overrightarrow{CB} = -5\overrightarrow{CP} \Leftrightarrow$

$-\overrightarrow{CP} - \overrightarrow{BC} = -5\overrightarrow{CP} \ / \cdot (-1) \Rightarrow \overrightarrow{CP} + \overrightarrow{BC} = 5\overrightarrow{C}$

$\Rightarrow 4\overrightarrow{CP} = \overrightarrow{BC} \Rightarrow \overrightarrow{CP} = \frac{1}{4}\overrightarrow{BC} \Leftrightarrow \overrightarrow{CP} = \frac{1}{4}(\overrightarrow{BA}$

$\Rightarrow \overrightarrow{CP} = \frac{1}{4}\left(-\overrightarrow{AB} + \overrightarrow{AC}\right) \Leftrightarrow \overrightarrow{CP} = -\frac{1}{4}\overrightarrow{AB} + \frac{1}{4}\overrightarrow{A}$

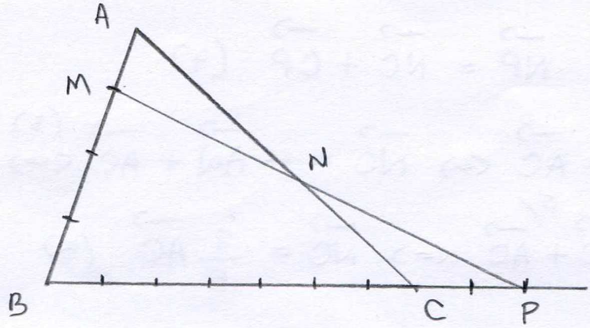
$,5,6 \Rightarrow \overrightarrow{NP} = \frac{3}{8}\overrightarrow{AC} - \frac{1}{4}\overrightarrow{AB} + \frac{\overset{2/}{1}}{4}\overrightarrow{AC} \Leftrightarrow$

$\overrightarrow{P} = -\frac{1}{4}\overrightarrow{AB} + \frac{5}{8}\overrightarrow{AC} \quad (**)$

$*, ** \Rightarrow \overrightarrow{MN} = \overrightarrow{NP} \Rightarrow \overrightarrow{MN}, \overrightarrow{NP}$ coliniari

$\Rightarrow M, N, P$ coliniare ■

5)



$(\vec{AB}, \vec{AC})$ - bază

În $\Delta AMN \overset{R.\Delta.}{\Rightarrow} \vec{MN} = \vec{MA} + \vec{AN}$ (1)

$\frac{MA}{MB} = \frac{1}{3} \Rightarrow \vec{MA} = -\frac{1}{3}\vec{MB} \Rightarrow \vec{MA} = -\frac{1}{3}(\vec{MA}$

$\Leftrightarrow \vec{MA} = -\frac{1}{3}\vec{MA} - \frac{1}{3}\vec{AB} \Leftrightarrow {}^{3/}\vec{MA} + \frac{1}{3}\vec{MA} = -$

$\Rightarrow \frac{4}{3}\vec{MA} = -\frac{1}{3}\vec{AB} \Big/ \cdot\frac{3}{4} \Rightarrow \vec{MA} = -\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{4}$

$\Rightarrow \vec{MA} = -\frac{1}{4}\vec{AB}$ (2)

$\frac{NC}{NA} = \frac{2}{3} \Rightarrow \vec{NA} = -\frac{3}{2}\vec{NC} \Leftrightarrow -\vec{AN} = -\frac{3}{2}(\vec{NA}$

$\Rightarrow \vec{AN} = \frac{3}{2}(-\vec{AN} + \vec{AC}) \Leftrightarrow \vec{AN} = -\frac{3}{2}\vec{AN} + \frac{3}{2}$

$\Leftrightarrow {}^{2/}\vec{AN} + \frac{3}{2}\vec{AN} = \frac{3}{2}\vec{AC} \Leftrightarrow \frac{5}{2}\vec{AN} = \frac{3}{2}\vec{AC} \Big/$

$\Rightarrow \vec{AN} = \frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{5}\vec{AC} \Rightarrow \vec{AN} = \frac{3}{5}\vec{AC}$ (3)

$1,2,3 \Rightarrow \vec{MN} = -\frac{1}{4}\vec{AB} + \frac{3}{5}\vec{AC}$

$\Delta NCP \overset{R.\Delta.}{\Rightarrow} \vec{NP} = \vec{NC} + \vec{CP}$ (4)

$\vec{NC} = \vec{NA} + \vec{AC} \Leftrightarrow \vec{NC} = -\vec{AN} + \vec{AC} \overset{(3)}{\Leftrightarrow}$

$\vec{NC} = -\frac{3}{5}\vec{AC} + {}^{5/}\vec{AC} \Leftrightarrow \vec{NC} = \frac{2}{5}\vec{AC}$ (5)

$\frac{PC}{PB} = \frac{2}{9} \Rightarrow \vec{CP} = -\frac{2}{9}\vec{PB} \Leftrightarrow \vec{CP} = -\frac{2}{9}(\vec{PC} + \vec{C}$

$\Rightarrow \vec{CP} = -\frac{2}{9}\vec{PC} - \frac{2}{9}\vec{CB} \Leftrightarrow \vec{CP} = \frac{2}{9}\vec{CP} + \frac{2}{9}\vec{BC}$

$\Rightarrow {}^{9/}\vec{CP} - \frac{2}{9}\vec{CP} = \frac{2}{9}(\vec{BA} + \vec{AC}) \Leftrightarrow$

$\Rightarrow \frac{7}{9}\vec{CP} = \frac{2}{9}(-\vec{AB} + \vec{AC}) \;/\cdot\frac{9}{7} \Rightarrow$

$\vec{CP} = \frac{2}{\cancel{9}}\cdot\frac{\cancel{9}}{7}(-\vec{AB} + \vec{AC}) \Leftrightarrow \vec{CP} = -\frac{2}{7}\vec{AB} + \frac{2}{7}$

$,5,6 \Rightarrow \vec{NP} = {}^{7/}\frac{2}{5}\vec{AC} - \frac{2}{7}\vec{AB} + {}^{5/}\frac{2}{7}\vec{AC} \Leftrightarrow$

$\vec{P} = -\frac{2}{7}\vec{AB} + \frac{24}{35}\vec{AC}$

$\vec{N} = \alpha\vec{NP} \Leftrightarrow -\frac{1}{4}\vec{AB} + \frac{3}{5}\vec{AC} = -\frac{2\alpha}{7}\vec{AB} + \frac{24\alpha}{35}\vec{AC}$

$\begin{cases} -\frac{2\alpha}{7} = -\frac{1}{4} \;/\cdot\left(-\frac{7}{2}\right) \\ \frac{24\alpha}{35} = \frac{3}{5} \;/\cdot\left(\frac{35}{24}\right) \end{cases} \Rightarrow \begin{cases} \alpha = \frac{7}{8} \\ \alpha = \frac{\cancel{3}}{\cancel{5}}\cdot\frac{\cancel{35}^{7}}{\cancel{24}_{8}} = \frac{7}{8} \end{cases}$

$\vec{N}, \vec{NP}$ colin $\Rightarrow$ M, N, P coliniare. ■